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The Method of Darboux

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1. INTRODUCTION

If a function $F(t)$ is regular at $t = 0$, then it has a Maclaurin expansion of the form

$$F(t) = \sum_{n=0}^{\infty} f_n t^n, \tag{1.1}$$

which will converge within the so-called circle of convergence, say $|t| < R$. If, as Darboux [1] assumed, R is finite, then $F(t)$ must have at least one singularity on the circle $|t| = R$. Darboux showed that if $F(t)$ had only a finite number of singularities on the circle of convergence, all of which were algebraic in nature, then the asymptotic behavior of f_n could be obtained as $n \to \infty$.

In this paper we shall investigate an extension of Darboux's result. We assume, as did Darboux, that on the circle of convergence $F(t)$ has only a finite number of singularities. Anticipating the final result, it is possible to assume a canonical form in which $F(t)$ has one and only one singularity on the circle of convergence, with the more general result being obtained by adding the contribution of each singularity. If the singularity occurs at $t = b$, then the substitution $t = bt'$ locates the singularity at $t' = 1$. Therefore, using the canonical form, it is assumed that $F(t)$ has a singularity at $t = 1$, and is regular within and on the contour C shown below. In a neighborhood of $t = 1$, $F(t)$ is assumed to have the form

$$F(t) = (1 - t)^{\lambda-1} (\log(1 - t))^{\mu} G(t) \tag{1.2}$$

where λ and μ are fixed complex numbers, $G(t)$ is regular at $t = 1$, and $\log(1 - t)$ has its principal value, which is real when t is real and <1. This certainly generalizes the Darboux condition in which the singularity of $F(t)$ at $t = 1$ was restricted to be of the form $(1 - t)^{\lambda-1}$.

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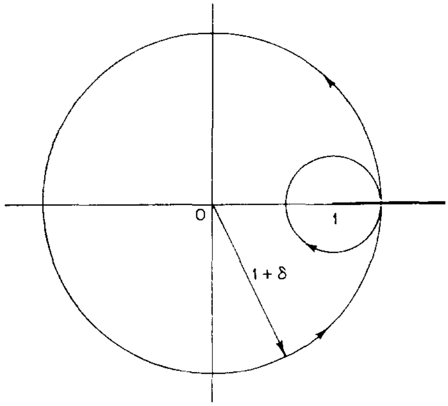

FIG. 1. Contour $C : \delta > 0$.

2. PRELIMINARIES

In this paper the asymptotic expansions obtained are not simply power series in n^{-1}. Instead they involve n in a more complicated manner and must be interpreted in the generalized sense of Erdélyi and Wyman [2].

Let $\{\varphi_m\}$ be an infinite sequence of functions $\varphi_m(n)$, where n is a large positive parameter. We say that $\{\varphi_m\}$ is an *asymptotic sequence* if, for all m,

$$\varphi_{m+1} = o(\varphi_m), \qquad \text{as} \quad n \to \infty. \tag{2.1}$$

Two functions $F(n)$ and $G(n)$ are said to be *asymptotically equal* relative to $\{\varphi_m\}$, written

$$F \approx G; \ \ \{\varphi_m\} \qquad \text{as} \quad n \to \infty, \tag{2.2}$$

if, for every fixed m

$$F = G + o(\varphi_m) \qquad \text{as} \quad n \to \infty. \tag{2.3}$$

The formal series $\sum F_m$ is said [2] to be an *asymptotic expansion* of the function F with respect to the asymptotic sequence $\{\varphi_m\}$, in symbols

$$F \sim \sum_{m=0}^{\infty} F_m \, ; \ \ \{\varphi_m\} \qquad \text{as} \quad n \to \infty, \tag{2.4}$$

if for every value of M,

$$F - \sum_{m=0}^{M} F_m = o(\varphi_M) \qquad \text{as} \quad n \to \infty. \tag{2.5}$$

Two functions having the same asymptotic expansion are asymptotically equal, and the converse is also true.

Returning to (1.1), we have by Cauchy's theorem

$$2\pi i f_n = \int_C t^{-n-1} F(t)\, dt,$$
$$= \int_{|t|=1+\delta} t^{-n-1} F(t)\, dt + \int_{|t-1|=\delta} t^{-n-1} F(t)\, dt. \qquad (2.6)$$

The path of integration on the large circle begins at $1 + \delta$, goes around the origin in the positive sense, and ends at $(1 + \delta)\, e^{2\pi i}$, whereas the path of integration on the small circle begins at $(1 + \delta)\, e^{2\pi i}$, goes around $t = 1$ in the negative sense, and ends at $1 + \delta$.

In (2.6), the number δ will no longer be considered to be fixed, and will be chosen to be

$$\delta = \delta_n = 1/n^{1/2}. \qquad (2.7)$$

On the circle $|t| = 1 + \delta_n$, we assume that $F(t)$ satisfies

$$F(t) = O(n^s), \qquad \text{as} \quad n \to \infty, \qquad (2.8)$$

for some fixed real number s. A simple estimation then gives

$$\int_{|t|=1+\delta_n} t^{-n-1} F(t)\, dt = O\left(\frac{n^s}{(1 + (1/n^{1/2}))^n}\right), \qquad \text{as} \quad n \to \infty,$$
$$= O(\exp(-\epsilon \sqrt{n})), \qquad \text{as} \quad n \to \infty,$$

for some fixed $\epsilon > 0$. This will imply

$$\int_{|t|=1+\delta_n} t^{-n-1} F(t)\, dt \approx 0; \qquad \{\varphi_m(n)\} \qquad \text{as} \quad n \to \infty, \qquad (2.10)$$

as long as $\{\varphi_n\}$ is any asymptotic sequence for which

$$\exp(-\epsilon n^{1/2}) \approx 0; \quad \{\varphi_m(n)\} \qquad \text{as} \quad n \to \infty. \qquad (2.11)$$

Anticipating the final result, it will be assumed that $\{\varphi_m\}$ is such an asymptotic sequence. Under these circumstances, we have

$$f_n \approx \frac{i}{2\pi} \int_{|t-1|=\delta_n} t^{-n-1} F(t)\, dt; \qquad \{\varphi_m(n)\} \qquad \text{as} \quad n \to \infty, \qquad (2.12)$$

where the path of integration on $|t - 1| = \delta_n$ is now orientated in the positive direction. The asymptotic behavior of f_n will hence be determined by the asymptotic behavior of the integral

$$I_n = \frac{i}{2\pi} \int_{|t-1|=\delta_n} t^{-n-1} F(t)\, dt. \tag{2.13}$$

Before we begin the study of the behavior of the integral I_n, we shall digress briefly to discuss the function

$$M(\lambda, \mu, n) = \frac{i}{2\pi} \int_{\infty}^{(0+)} (-t)^{\lambda-1} (\log(-t))^{\mu} e^{-(n+1)t}\, dt, \tag{2.14}$$

where the loop contour of integration and the cuts in the t-plane are illustrated below. By a modification of a result proved in [6], we can show that

$$M(\lambda, \mu, n) \sim \frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda}} \left[\sum_{k=0}^{\infty} \binom{\mu}{k} \frac{D^k[\Gamma^{-1}(1-\lambda)]}{(-\log(n+1))^k} ; \{(\log(n+1))^{-k}\}\right] \tag{2.15}$$

as $n \to \infty$, where $D^k = d^k/d\lambda^k$.

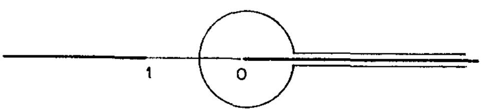

FIG. 2. t-plane.

This result is ultimately used in a slightly different form. Let $P_m(\omega)$ be the polynomials defined by

$$G(t+1) \exp\left\{-\frac{1}{2}\,\omega t \left[\frac{2(\log(t+1) - t)}{t^2}\right]\right\} = \sum_{m=0}^{\infty} P_m(\omega)\, t^m, \tag{2.16}$$

where $G(t)$ is the function given in (1.2).

An explicit expression for $P_m(\omega)$ is given by

$$P_m(\omega) = \frac{1}{m!} \frac{d^m}{dt^m} \left[G(t+1) \exp\left\{-\frac{1}{2}\,\omega t \left[\frac{2(\log(t+1) - t)}{t^2}\right]\right\}\right]_{t=0}. \tag{2.17}$$

Consider the integral

$$J_m = \frac{i}{2\pi} \int_{\gamma_n} (-t)^{\lambda+m-1} (\log(-t))^{\mu} P_m((n+1)\, t)\, e^{-(n+1)t}\, dt, \tag{2.18}$$

where γ_n is the path of integration which traverses on the circle $|t| = \delta_n$ in the positive direction, and begins and ends on the positive half of the real axis. The polynomials $P_m((n+1)t)$ may be written

$$P_m((n+1)\,t) = \sum_{s=0}^{m} p_s(n+1)^s\, t^s \tag{2.19}$$

where p_s is a fixed number. Hence

$$J_m = \sum_{s=0}^{m} p_s(n+1)^s \frac{i}{2\pi} \int_{\gamma_n} (-t)^{\lambda+m+s-1} (\log(-t))^{\mu}\, e^{-(n+1)t}\, dt. \tag{2.20}$$

Since the error incurred by extending the circular paths of integration to infinite loops is $O(\exp(-\epsilon n^{1/2}))$, (cf. [6, (2.8)]), we have in view of (2.11)

$$J_m \approx \sum_{s=0}^{m} p_s(n+1)^s\, M(\lambda + m + s, \mu, n); \qquad \{\varphi_m(n)\} \tag{2.21}$$

as $n \to \infty$, and hence by (2.15)

$$J_m \sim \frac{(-\log(n+1))^{\mu}}{(n+1)^{\lambda+m}} \left[\sum_{k=0}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} A_k(\lambda, m); \{(\log(n+1))^{-k}\}\right] \tag{2.22}$$

as $n \to \infty$, where

$$A_k(\lambda, m) = \sum_{s=0}^{m} p_s D^k[\Gamma^{-1}(1 - \lambda - m - s)]. \tag{2.23}$$

3. Asymptotic Expansion of f_n

Returning to (2.13), $(t-1)$ is replaced by t to obtain

$$I_n = \frac{i}{2\pi} \int_{\gamma_n} (t+1)^{-n-1} F(t+1)\, dt, \tag{3.1}$$

where the path of integration γ_n is described in (2.18).

Theorem. *If $F(t)$ is regular within and on the contour C shown in Fig.* 1, *and if $F(t)$ satisfies the conditions* (1.2) *and* (2.8), *then*

$$f_n \sim \sum_{m=0}^{\infty} (-1)^m J_m\,; \qquad \left\{\frac{(\log n)^{\mu}}{n^{\lambda+m}}\right\} \qquad as \quad n \to \infty, \tag{3.2}$$

where the J_m are functions of n given by (2.18).

Proof. Substitution of (1.2) into (3.1) yields

$$I_n = \frac{i}{2\pi} \int_{\gamma_n} (-t)^{\lambda-1} (\log(-t))^{\mu}\, G(t+1)(t+1)^{-n-1}\, dt. \tag{3.3}$$

The factor $G(t+1)\exp\{-(n+1)[\log(t+1)-t]\}$ is written as

$$\begin{aligned} &G(t+1)\exp\{-(n+1)[\log(t+1)-t]\} \\ &\quad = G(t+1)\exp\left\{-\frac{1}{2}\,\omega t\left[\frac{2(\log(t+1)-t)}{t^2}\right]\right\} \end{aligned} \tag{3.4}$$

where

$$\omega = (n+1)t. \tag{3.5}$$

The expression on the right side of (3.4) will have the convergent expansion (2.16) as long as

$$\omega t = O(1), \qquad \text{as} \quad n \to \infty. \tag{3.6}$$

Since $\omega t = (n+1)\,t^2$, $\omega t = O(1)$, as $n \to \infty$, will certainly be satisfied within and on $|\,t\,| = K/n^{1/2}$, where K is any fixed positive number. Hence, for any fixed integer $N \geqslant 0$, (2.16) can be written as

$$G(t+1)\exp\left\{-\frac{1}{2}\,\omega t\left[\frac{2(\log(t+1)-t)}{t^2}\right]\right\} = \sum_{m=0}^{N} P_m(\omega)\, t^m + R_N \tag{3.7}$$

where R_N is regular at $t = 0$, and for which

$$R_N = O(n^{(N+1)/2} t^{N+1}), \qquad \text{as} \quad n \to \infty, \tag{3.8}$$

providing $|\,t\,| \leqslant K/n^{1/2}$. Coupling the results (3.4) and (3.7) together gives

$$I_n = \sum_{m=0}^{N} (-1)^m J_m + E_N \tag{3.9}$$

where J_m is given by (2.18) and

$$E_N = \frac{i}{2\pi} \int_{\gamma_n} (-t)^{\lambda-1} (\log(-t))^{\mu}\, R_N e^{-(n+1)t}\, dt. \tag{3.10}$$

It is always possible to choose the integer N large enough so that $\mathrm{Re}(\lambda + N + 1) > 0$, and therefore the regularity of the integrand will allow the replacement of the circular path of integration by two straight lines joining $t = 0$ to $t = \delta$, one on the top side of the cut in the t-plane, and the other on the lower side of this cut.

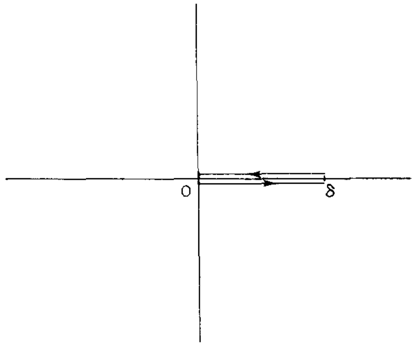

FIG. 3. The path of integration L.

Hence

$$| E_N | = O\left(n^{(N+1)/2} \int_L |(-t)^{\lambda+N} (\log(-t))^{\mu} e^{-(n+1)t} dt |\right), \quad \text{as} \quad n \to \infty, \tag{3.11}$$

and, by Lemma 4.1 in [5],

$$| E_N | = O((\log n)^{\mu}/n^{\lambda+(N+1)/2}), \quad \text{as} \quad n \to \infty. \tag{3.12}$$

Since

$$\varphi_m = (\log n)^{\mu}/n^{\lambda+m/2} \tag{3.13}$$

is an asymptotic sequence, in view of (2.12)

$$\begin{aligned} f_n &\approx I_n \quad ; \quad \{(\log n)^{\mu}/n^{\lambda+m/2}\} \\ f_n &\sim \sum_{m=0}^{\infty} (-1)^m J_m \; ; \quad \{(\log n)^{\mu}/n^{\lambda+m/2}\} \end{aligned} \tag{3.14}$$

as $n \to \infty$.

The order of the terms J_m, given in (2.22), indicates that the result in (3.14) can be improved to read

$$f_n \sim \sum_{m=0}^{\infty} (-1)^m J_m \; ; \quad \left\{\frac{(\log n)^{\mu}}{n^{\lambda+m}}\right\} \quad \text{as} \quad n \to \infty, \tag{3.15}$$

which is identical with the statement of the theorem.

Remark 1. When $\mu = 0$, the canonical form (1.2) reduces to the Darboux condition and our asymptotic expansion (3.2) is equivalent to a recent result given in Erdélyi and Wyman [2]. To some extent our analysis is based on [2].

Remark 2. When μ is a positive integer and $G(t) = 1$, the asymptotic behavior of f_n was investigated by S. Narumi [4] who found the dominant term of an asymptotic expansion with an error term. Narumi's method divides the asymptotic formula into two cases according as λ is a nonnegative integer or not a nonnegative integer. Our method makes no such separation and the two cases are treated as one.

4. Comments and Extensions

From (2.22),

$$J_0 \sim \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} \Big[\sum_{k=0}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} p_0 D^k[\Gamma^{-1}(1-\lambda)] ; \{(\log(n+1))^{-k}\} \Big], \quad \text{as} \quad n \to \infty, \tag{4.1}$$

and

$$J_1 \sim \frac{(-\log(n+1))^\mu}{(n+1)^{\lambda+1}} \Big[\sum_{k=0}^{\infty} \binom{\mu}{k} (-\log(n+1))^{-k} \sum_{s=0}^{1} p_s D^k[\Gamma^{-1}(-\lambda-s)]; \{(\log(n+1))^{-k}\} \Big], \quad \text{as} \quad n \to \infty. \tag{4.2}$$

Hence, for any integer $N \geqslant 0$

$$J_0 - J_1 = G(1) \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} \Big[\sum_{k=0}^{N} \binom{\mu}{k} (-\log(n+1))^{-k} D^k[\Gamma^{-1}(1-\lambda)] + O((\log(n+1))^{-N-1}) + O(1/n) \Big] \tag{4.3}$$

as $n \to \infty$. Clearly, none of the terms of J_1 can contribute to the asymptotic expansion for f_n unless the infinite asymptotic expansion for J_0 terminates after a finite number of terms. The same will obviously be true for J_m, $m \geqslant 1$. Hence the general situation is

$$f_n \sim G(1) \frac{(-\log(n+1))^\mu}{(n+1)^\lambda} \Big[\sum_{k=0}^{\infty} \binom{\mu}{k} \frac{D^k[\Gamma^{-1}(1-\lambda)]}{(-\log(n+1))^k} ; \{(\log(n+1))^{-k}\} \Big] \tag{4.4}$$

as $n \to \infty$.

For the special case μ a nonnegative integer, a more accurate asymptotic expansion does exist. This is true because the infinite series expansions for J_m all terminate. Returning to (2.18), the path of integration γ_n can be replaced by the infinite loop with the introduction of a term that is exponentially small. Hence

$$
\begin{aligned}
J_m &\approx \frac{i}{2\pi} \int_\infty^{(0+)} (-t)^{\lambda+m-1} (\log(-t))^\mu P_m((n+1)t)\, e^{-(n+1)t}\, dt \\
&\approx \frac{i}{2\pi} \frac{d^\mu}{d\lambda^\mu} \int_\infty^{(0+)} (-t)^{\lambda+m-1} P_m((n+1)t)\, e^{-(n+1)t}\, dt \qquad (4.5) \\
&\approx \frac{i}{2\pi} \frac{d^\mu}{d\lambda^\mu} \left[(n+1)^{-\lambda-m} \int_\infty^{(0+)} (-\omega)^{\lambda+m-1} P_m(\omega)\, e^{-\omega}\, d\omega\right];\ \{\varphi_m(n)\}.
\end{aligned}
$$

From (2.17), (4.5) can be written

$$
\begin{aligned}
J_m &\approx \frac{i}{2\pi} \frac{d^\mu}{d\lambda^\mu} \left[\frac{1}{m!\,(n+1)^{\lambda+m}} \cdot \frac{d^m}{dt^m} \left\{G(t+1) \int_\infty^{(0+)} (-\omega)^{\lambda+m-1}\right.\right. \\
&\qquad \left.\left. \cdot \exp\left[-\omega \left(\frac{\log(t+1)}{t}\right)\right] d\omega\right\}\right] \qquad (4.6) \\
&\approx \frac{1}{m!} \frac{d^\mu}{d\lambda^\mu} \left[\frac{d^m}{dt^m} \left\{G(t+1)\left[\frac{t}{\log(t+1)}\right]^{\lambda+m}\right\}_{t=0} \left(\frac{(n+1)^{-\lambda-m}}{\Gamma(1-\lambda-m)}\right)\right].
\end{aligned}
$$

The general form of (4.6) is

$$J_m \approx [(-\log(n+1))^\mu/(n+1)^{\lambda+m}]\, Q_\mu((\log(n+1))^{-1}), \qquad (4.7)$$

where $Q_\mu(z)$ is a polynomial whose degree does not exceed μ. In (3.2), there is therefore no need to drop any of the terms, and the more accurate expansion is worth retaining.

The results thus far obtained, derived for a so-called canonical form, allow a direct derivation of a more general result. Returning to (1.2), it will now be assumed

$$F(t+1) \sim \sum_{m\in I} (-t)^{\lambda_m-1} (\log(-t))^{\mu_m} G_m(t); \qquad \{(-t)^{\lambda_m-1} (\log(-t))^{\mu_m}\}, \qquad (4.8)$$

as $t \to 0$ in $-\pi \leqslant \arg(-t) \leqslant \pi$, where I stands for either the finite set of integers $\{0, 1, 2,..., M\}$ or the infinite set of integers $\{0, 1, 2,...\}$. In (4.8), each $G_m(t)$ is regular at $t = 0$. Hence for each fixed $N \in I$,

$$F(t+1) = \sum_{m=0}^{N} (-t)^{\lambda_m-1} (\log(-t))^{\mu_m} G_m(t) + R_N(t), \qquad (4.9)$$

where

$$R_N(t) = o((-t)^{\lambda_N-1} (\log(-t))^{\mu_N}), \qquad \text{as} \quad t \to 0. \qquad (4.10)$$

Hence, I_n of (3.1) becomes

$$I_n = \sum_{m=0}^{N} \int_{\gamma_n} (-t)^{\lambda_m - 1} (\log(-t))^{\mu_m} G_m(t)(t+1)^{-n-1}\, dt$$
$$+ \int_{\gamma_n} R_N(t)(t+1)^{-n-1}\, dt. \tag{4.11}$$

As long as an integer N can be chosen so that

$$\operatorname{Re} \lambda_N > 0, \tag{4.12}$$

and $R_N(t)$ is sufficiently regular in the cut neighborhood $|t| \leqslant \delta_n$ to replace the circular path of integration by the straight line segments joining $t = 0$ to $t = \delta_n$, above and below the cut, then

$$\int_{\gamma_n} R_N(t)(t+1)^{-n-1}\, dt = O\left(\frac{(\log(n+1))^{\mu_N}}{(n+1)^{\lambda_N}}\right), \quad \text{as } n \to \infty. \tag{4.13}$$

Finally, each term of the asymptotic expansion is of the canonical form discussed in Section 3, and the asymptotic behavior of each term can be determined as $n \to \infty$.

The choice of the factor $(-t)^{\lambda_m}$ rather than t^{λ_m} in our general result was dictated by a desire to provide direct application of (2.15). Clearly if the natural factors in the expansion of $F(t+1)$ have the form $t^{\lambda_m}(\log t)^{\mu_m}$, it would be possible to write $t = e^{ik\pi}(-t)$, $t^{\lambda_m} = e^{ik\pi\lambda_m}(-t)^{\lambda_m}$, and $\log t = \log(-t) + ik\pi$, for some integer k. This would imply an awkward reexpansion in order that a direct application of the results of this paper be applicable to the factor $t^{\lambda_m}(\log t)^{\mu_m}$. Rather than follow this course, it is recommended that one recast the results of the paper using

$$\int_{\infty}^{(0+)} t^{\lambda-1} e^{-zt}\, dt = \frac{2\pi i}{(ze^{-i\pi})^{\lambda}\, \Gamma(1-\lambda)} \tag{4.14}$$

as the basic integral. For any positive integer μ, one obtains by differentiating μ times with respect to λ

$$\frac{1}{2\pi i} \int_{\infty}^{(0+)} t^{\lambda-1} (\log t)^{\mu}\, e^{-zt}\, dt = \sum_{r=0}^{\mu} \binom{\mu}{r} D^r[\Gamma^{-1}(1-\lambda)]\, D^{\mu-r}[(ze^{-i\pi})^{-\lambda}], \tag{4.15}$$

and the corresponding exact expression when μ is a nonnegative integer is replaced by an asymptotic expansion for all other values of μ. Obviously everything can then be repeated to derive the analogous result when terms of the form $t^{\lambda-1}(\log t)^{\mu}$ are the natural factors to use in the expansion of $F(t+1)$.

5. Example

A typical example of a function which satisfies the conditions of our canonical form (1.2) is provided by

$$F(t) = (1 - t)^{\lambda-1} [\log(1 - t)/t]^{\mu}, \qquad F(0) = e^{i\pi\mu}, \tag{5.1}$$

where λ and μ are fixed complex numbers.

However, only the special case

$$\begin{aligned} F(t) &= [\log(1 - t)/t]^{\mu}, \\ &= \sum_{n=0}^{\infty} A_n^{(\mu)} t^n, \qquad A_0^{(\mu)} = e^{i\pi\mu}, \end{aligned} \tag{5.2}$$

will be used to compare our procedure with known methods of finding the asymptotic behavior of Stirling numbers of the first kind. It will also be used to illustrate that the final form of an asymptotic expansion so often depends on the procedure which is used to develop the form.

As our general theorem shows

$$\begin{aligned} A_n^{(\mu)} &\approx \frac{i}{2\pi} \int_{|t-1|=\delta_n} t^{-n-1} F(t)\, dt \\ &\approx \frac{i}{2\pi} \int_{\gamma_n} (t + 1)^{-n-1} F(t + 1)\, dt. \end{aligned} \tag{5.3}$$

In applying our previous results, $\lambda = 1$, $G(t) = (1 + t)^{-\mu}$, and the asymptotic expansion can be immediately obtained from (3.2). It is simpler however to write

$$A_n^{(\mu)} \approx \frac{i}{2\pi} \int_{|t|=\delta_n} [\log(-t)]^{\mu} (t + 1)^{-n-\mu-1}\, dt, \tag{5.4}$$

and consider $\lambda = 1$, $G(t) = 1$ and $z = n + \mu + 1$ as the asymptotic variable. The results will apply even though μ may be a complex number.

Writing

$$A_n^{(\mu)} \approx \frac{i}{2\pi} \int_{\gamma_n} [\log(-t)]^{\mu} \exp\left[-\frac{1}{2}\, \omega t \left\{\frac{2(\log(t + 1) - t)}{t^2}\right\}\right] e^{-(n+\mu+1)t}\, dt \tag{5.5}$$

where $\omega = (n + \mu + 1)t$, we have seen that only the first term in the expansion of

$$\exp\left[-\frac{1}{2}\, \omega t \left\{\frac{2(\log(t + 1) - t)}{t^2}\right\}\right] = \sum_{m=0}^{\infty} P_m(\omega)\, t^m \tag{5.6}$$

need be considered as long as μ is not a nonnegative integer. In this case

$$A_n^{(\mu)} \approx \frac{i}{2\pi} \int_{\gamma_n} (\log(-t))^\mu e^{-(n+\mu+1)t}\, dt$$

$$\sim \frac{e^{i\pi\mu}[\log(n+\mu+1)]^\mu}{(n+\mu+1)} \Big[\sum_{k=0}^{\infty} \binom{\mu}{k} D^k \Big[\Gamma(\lambda) \frac{\sin \pi\lambda}{\pi}\Big]_{\lambda=1}$$

$$\cdot (-\log(n+\mu+1))^{-k}; \qquad \{(\log(n+\mu+1))^{-k}\}\Big] \tag{5.7}$$

as $n \to \infty$. The first two nonzero terms of (5.7) are given by

$$A_n^{(\mu)} = \frac{e^{i\pi\mu}[\log(n+\mu+1)]^\mu}{(n+\mu+1)} \Big[\frac{\mu}{\log(n+\mu+1)} - \frac{\mu(\mu-1)\,\Gamma'(1)}{(\log(n+\mu+1))^2}$$

$$+ O\Big(\frac{1}{(\log(n+\mu+1))^3}\Big)\Big]$$

$$= \frac{\mu e^{i\pi\mu}}{(n+\mu+1)} (\log(n+\mu+1))^{\mu-1}$$

$$\times \Big[1 - \frac{(\mu-1)\,\Gamma'(1)}{\log(n+\mu+1)} + O\Big(\frac{1}{(\log(n+\mu+1))^2}\Big)\Big] \tag{5.8}$$

as $n \to \infty$.

The stirling numbers of the first kind S_n^m have the generating function

$$[\log(1+t)]^m = \sum_{n=m}^{\infty} \frac{m!}{n!} S_n^m t^n \tag{5.9}$$

with the obvious relation to the $A_n^{(\mu)}$ given above by

$$A_n^{(\mu)} = (\mu!/(n+\mu)!)\, S_{n+\mu}^{\mu} (-1)^{\mu+n}, \tag{5.10}$$

when μ is a nonnegative integer.

Jordan [6, p. 161], gives the asymptotic result

$$|\, S_{n+\mu}^{\mu} \,| \sim ((n+\mu-1)!/(\mu-1)!)[\log(n+\mu)+\gamma]^{\mu-1}, \tag{5.11}$$

where $\gamma = -\Gamma'(1)$ is Euler's constant. The two results (5.8) and (5.11) agree to the order indicated in (5.8). However, the procedures of the present paper allow a much deeper result to be obtained when μ is not a nonnegative integer.

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